Material Forces in the Context of Biological Tissue Remodelling

H. Narayanan, K. Garikipati, E. M. Arruda, K. Grosh

University of Michigan, Ann Arbor

Seventh U.S. National Congress on Computational Mechanics

Albuquerque, New Mexico, July 27–31, 2003

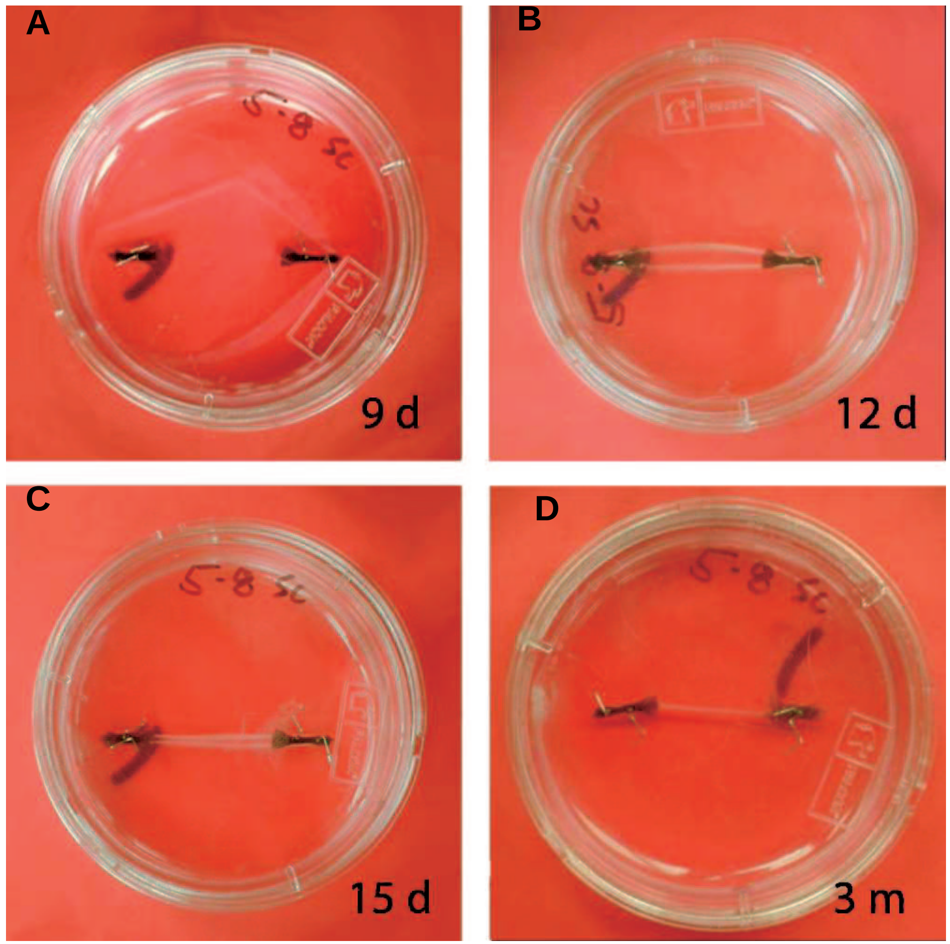
Development of Biological Tissue

Growth and Remodelling

- Growth is a change in density due to mass transport (Epstein & Maugin [2000], Tao et al. [2001], Taber & Humphrey [2001], Humphrey & Rajagopal [2002], Lubarda & Hoger [2002], Kuhl & Steinmann [2002], KG et al. [2003])
  - Tissue is open with respect to mass
  - Multiple species, treated by mixture theory
- Remodelling is an evolution of the microstructure (Taber & Humphrey [2001], Ambrosi & Mollica [2002], Humphrey & Rajagopal [2002])
  - Local reconfiguration of material: self-assembly
  - Evolution of “reference” configuration: *remodelled configuration*

Development of Biological Tissue

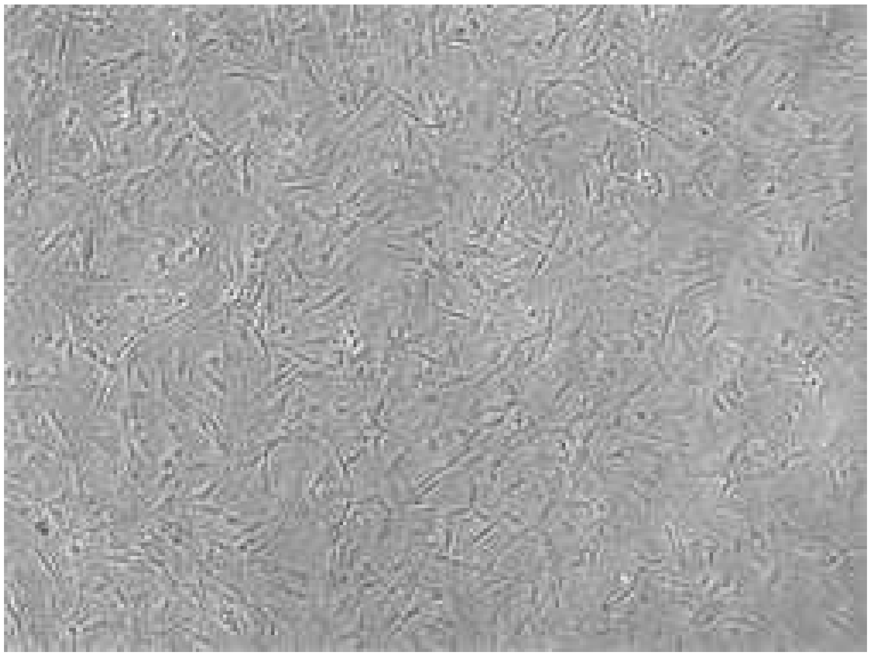
Growth of tendon constructs

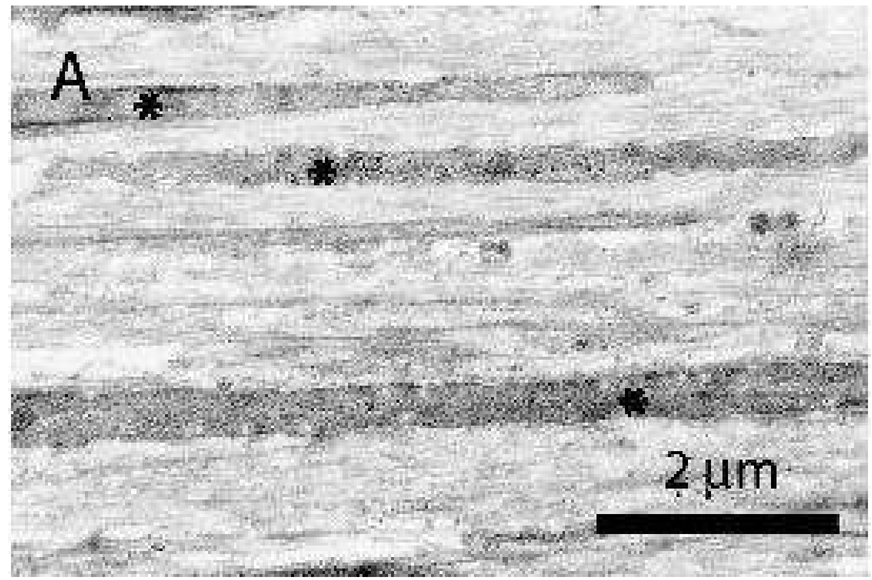


Calve et al. 2003

Development of Biological Tissue

Remodelling of collagen during growth

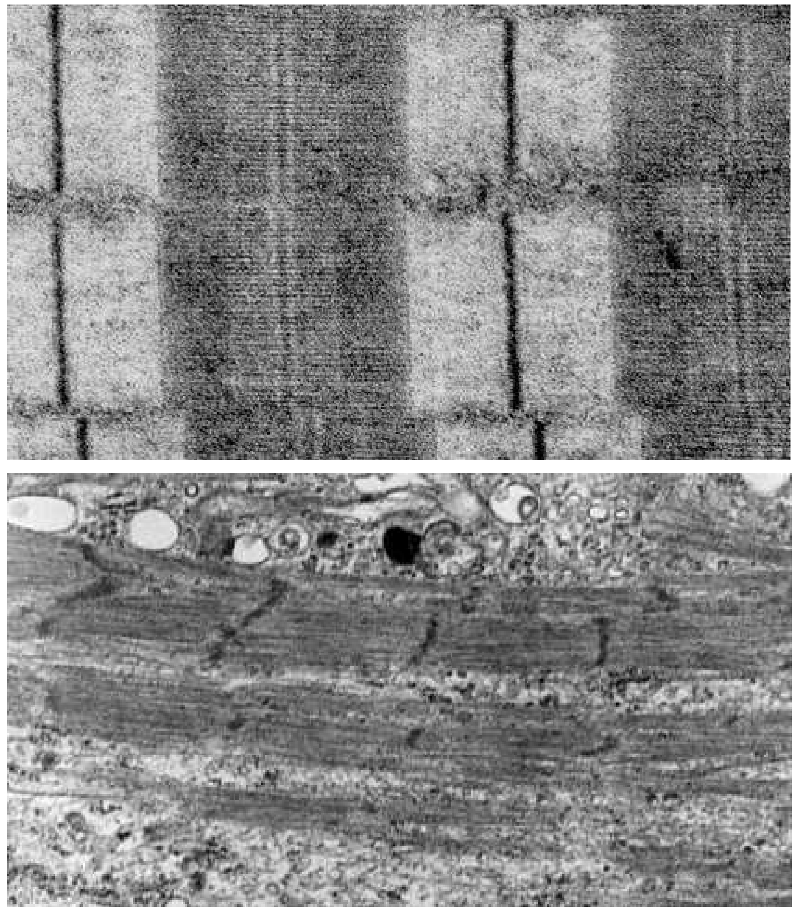




Calve et al. 2003

Development of Biological Tissue

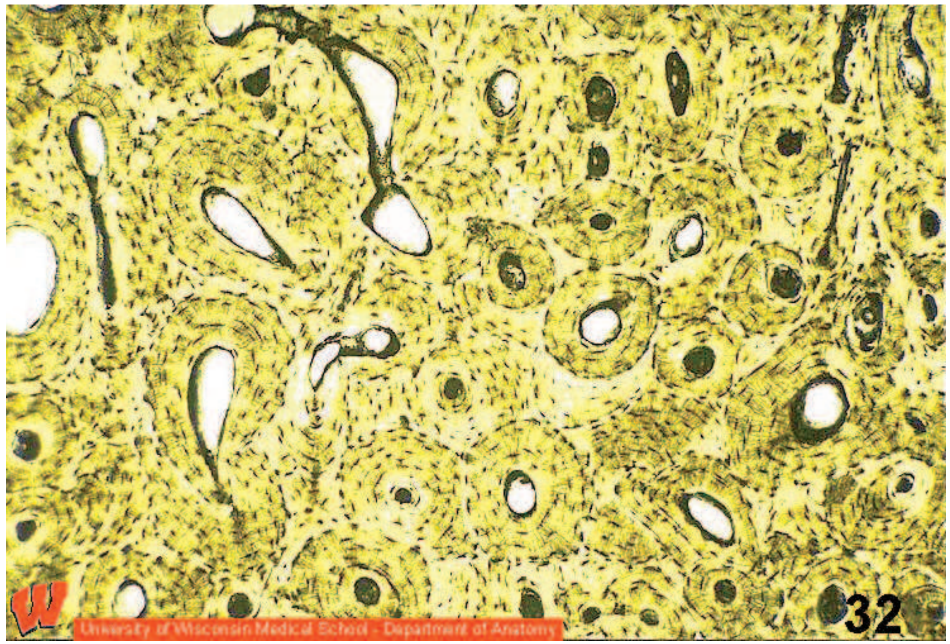
Remodelling during growth



Hirsch et al. 1998

Development of Biological Tissue

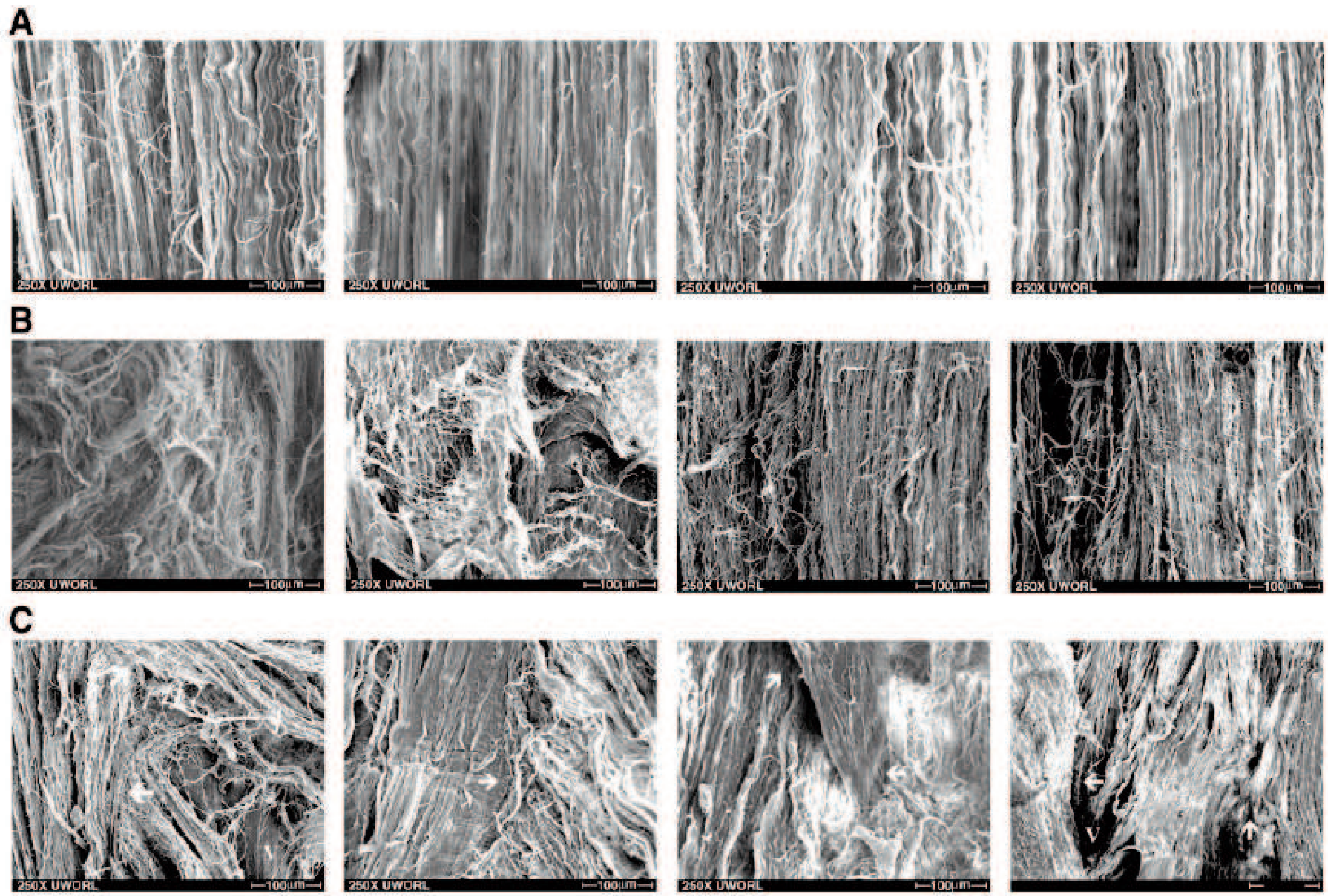
Remodelling of bone



- University of Wisconsin, Dept. of Anatomy
- The tissue reconfigures by changing its microstructure when stressed (Wolff [1892])

Development of Biological Tissue

Remodelling of collagen due to load while healing

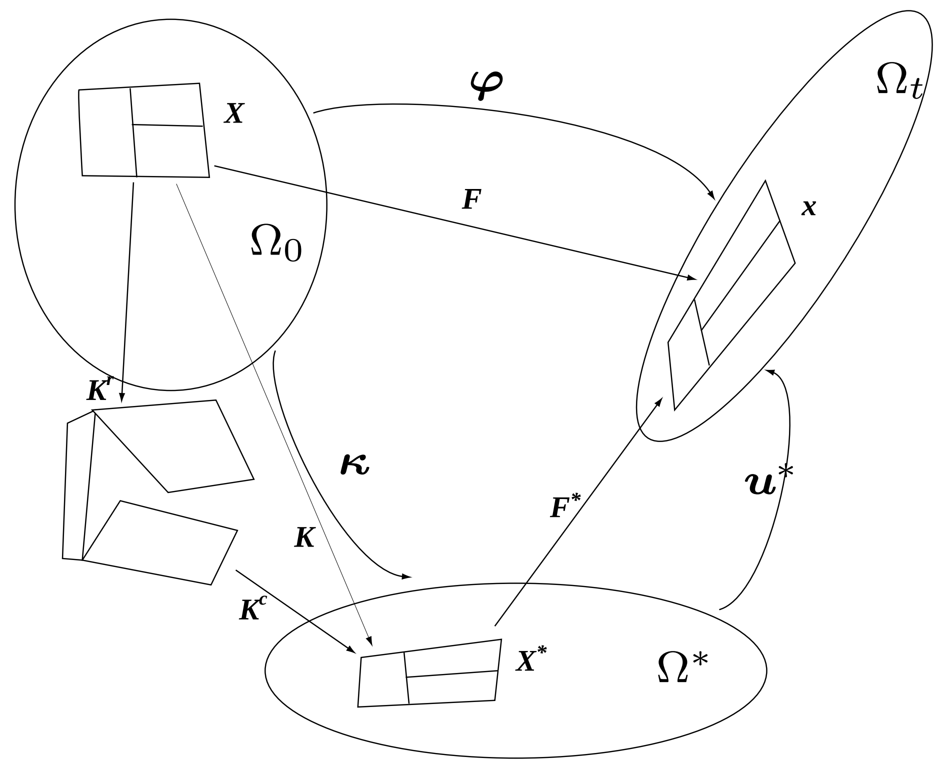


Provenzano et al. 2003

Development of Biological Tissue

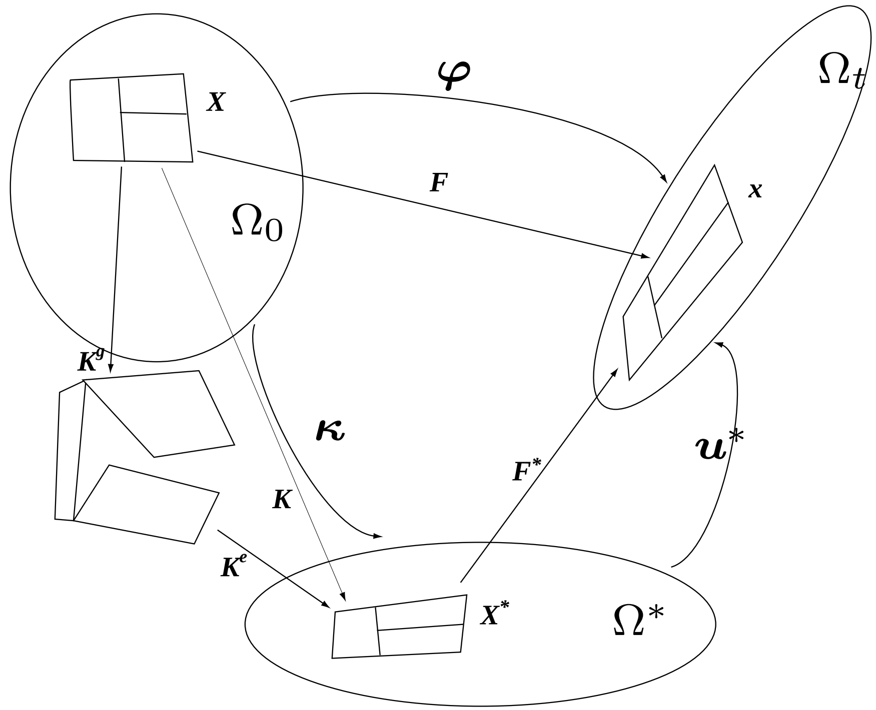
- Remodelling is the reconfiguration of the material
  - Stress-driven
  - “Preferred” configuration that varies pointwise and is in general incompatible. A further configurational change can occur, resulting in a compatible configuration.
- Biological tissue is capable of changes in configuration by motion of particles relative to ambient material
  - *Motion in material space/Configurational change*

Continuum Field Formulation



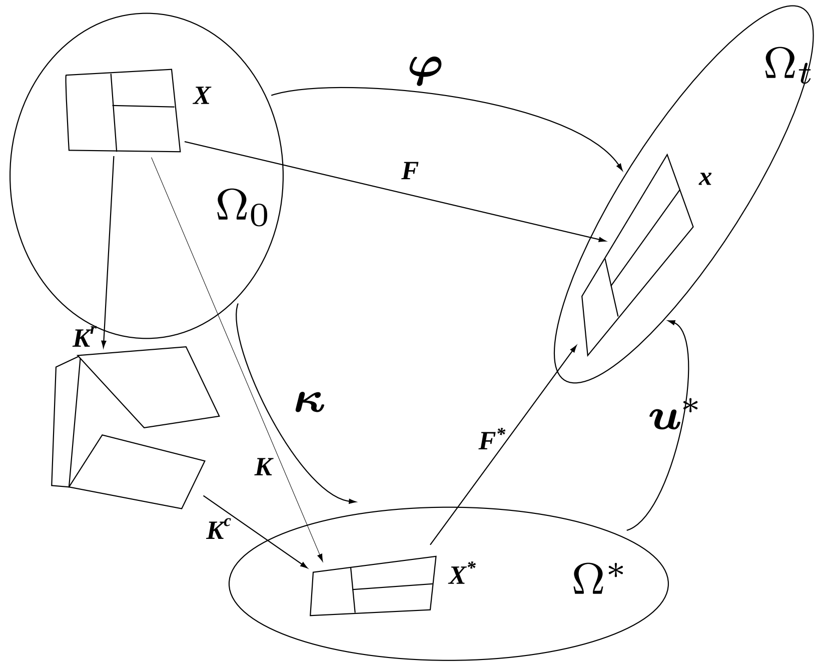
- $\boldsymbol{K}^{\mathrm{r}}$ is given. $\boldsymbol{\kappa}(\boldsymbol{X}, t) = ?$ (motion in material space)

Continuum Field Formulation



- $\boldsymbol{K}^{\mathrm{g}}$ is a kinematic “growth” tensor , $\boldsymbol{K}^{\mathrm{e}}$ and $\boldsymbol{F}^*$ are elastic deformation gradients—internal stress problem

A Variational Method



$$\Pi[\boldsymbol{u}^*, \boldsymbol{\kappa}] := \int\limits_{\Omega^*} \hat{\psi}^*(\boldsymbol{F}^*, \boldsymbol{K}^{\mathrm{c}}, \boldsymbol{X}^*)\mathrm{d}V^* - \int\limits_{\Omega^*} \boldsymbol{f}^* \cdot (\boldsymbol{u}^* + \boldsymbol{\kappa})\mathrm{d}V^* - \int\limits_{\partial\Omega^*} \boldsymbol{t}^* \cdot (\boldsymbol{u}^* + \boldsymbol{\kappa})\mathrm{d}A^*$$

A Variational Method

- Variation in spatial position: $\boldsymbol{u}^*_\varepsilon = \boldsymbol{u}^* + \varepsilon\delta\boldsymbol{u}^*$
- Equilibrium with respect to $\boldsymbol{u}^*$:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Pi[\boldsymbol{u}^*_\varepsilon, \boldsymbol{\kappa}]\Big|_{\varepsilon=0} = 0$$

- Euler-Lagrange equations:

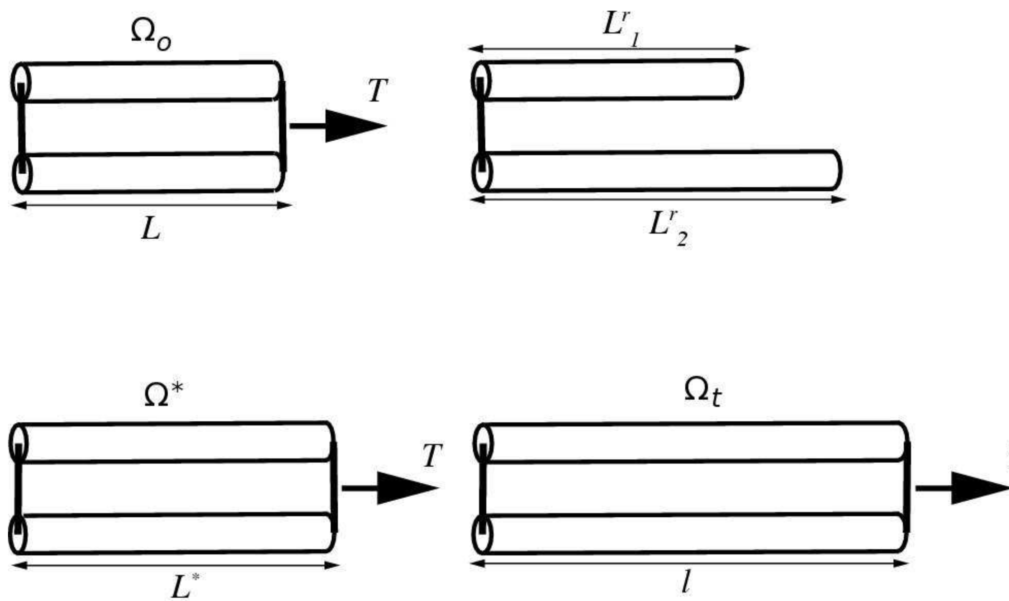
$$\mathrm{Div}^*\boldsymbol{P}^* + \boldsymbol{f}^* = \mathbf{0}, \text{ in } \Omega^*; \quad \boldsymbol{P}^*\boldsymbol{N}^* = \boldsymbol{t}^* \text{ on } \partial\Omega^*; \text{ where } \boldsymbol{P}^* := \frac{\partial\psi^*}{\partial\boldsymbol{F}^*}$$

- Quasistatic balance of linear momentum in remodelled configuration, Ω^*

A Variational Method

- Equilibrium with respect to material motion:
  $\boldsymbol{\kappa}_\varepsilon = \boldsymbol{\kappa} + \varepsilon\delta\boldsymbol{\kappa}$
  $$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Pi[\boldsymbol{u}^*, \boldsymbol{\kappa}_\varepsilon]\bigg|_{\varepsilon=0} = 0$$
- Euler-Lagrange equations:
  $$-\mathrm{Div}^*(\psi^*\mathbf{1} - \boldsymbol{F}^{*\mathrm{T}}\boldsymbol{P}^* + \boldsymbol{\Sigma}^*) + \frac{\partial\psi^*}{\partial\boldsymbol{X}^*} = \mathbf{0} \text{ in } \boldsymbol{\Omega}^*,$$
  $$-\left(\psi^*\mathbf{1} - \boldsymbol{F}^{*\mathrm{T}}\boldsymbol{P}^* + \boldsymbol{\Sigma}^*\right)\boldsymbol{N}^* = \mathbf{0} \text{ on } \partial\Omega^*$$
  $$\text{where } \boldsymbol{\Sigma}^* := \frac{\partial\psi^*}{\partial\boldsymbol{K}^{\mathrm{c}}}\boldsymbol{K}^{\mathrm{cT}}$$
- Eshelby stress: $\psi^*\mathbf{1} - \boldsymbol{F}^{*\mathrm{T}}\boldsymbol{P}^*$; configurational stress: $\boldsymbol{\Sigma}^*$

Remodelling Examples

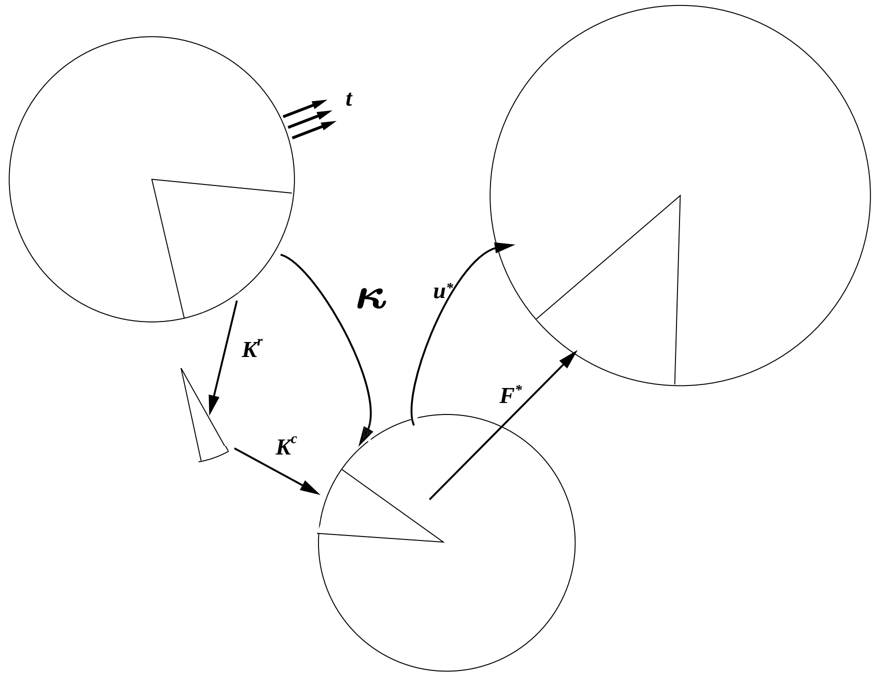


$$\kappa = L^* - L, \quad u^* = l - L^*$$

$$\Pi[u^*, \kappa] = \frac{1}{2}k^*(\kappa + L - L_1^{\mathrm{r}})^2 + \frac{1}{2}k^*(\kappa + L - L_2^{\mathrm{r}})^2 + 2 \cdot \frac{1}{2}k{u^*}^2 - T(u^* + \kappa)$$

$$\frac{\partial \Pi}{\partial u^*} = 0 \quad \Rightarrow \; 2ku^* = T; \qquad \frac{\partial \Pi}{\partial \kappa} = 0 \quad \Rightarrow \; \boldsymbol{\kappa} = \frac{k}{k^*}u^* - \left(L - \frac{L_1^{\mathrm{r}} + L_2^{\mathrm{r}}}{2}\right)$$

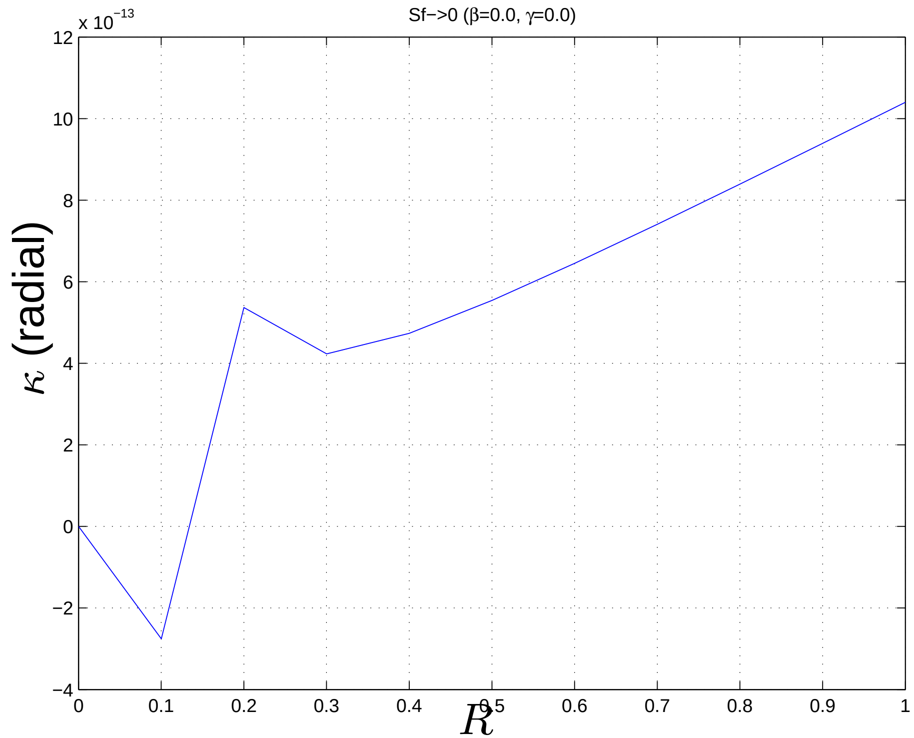
Remodelling Examples



$$\boldsymbol{K}^{\mathrm{r}} = \begin{bmatrix} 1+\beta & 0 & 0 \\ 0 & 1+\gamma & 0 \\ 0 & 0 & 1+\gamma \end{bmatrix}, \quad \boldsymbol{t}^* = \delta \boldsymbol{e}_R$$

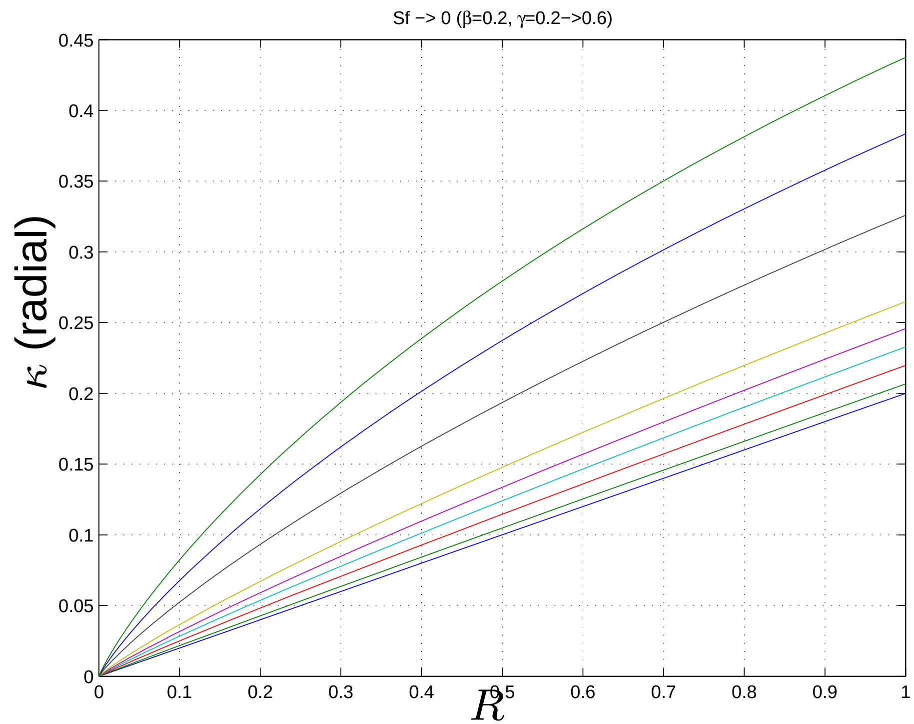
- $\hat{\psi}^*(\boldsymbol{F}^*, \boldsymbol{K}^{\mathrm{c}}, \boldsymbol{X}^*) = \hat{\psi}_1^*(\boldsymbol{F}^*) + \hat{\psi}_2^*(\boldsymbol{K}^{\mathrm{c}})$, (compressible neo-Hookean)

Remodelling Examples



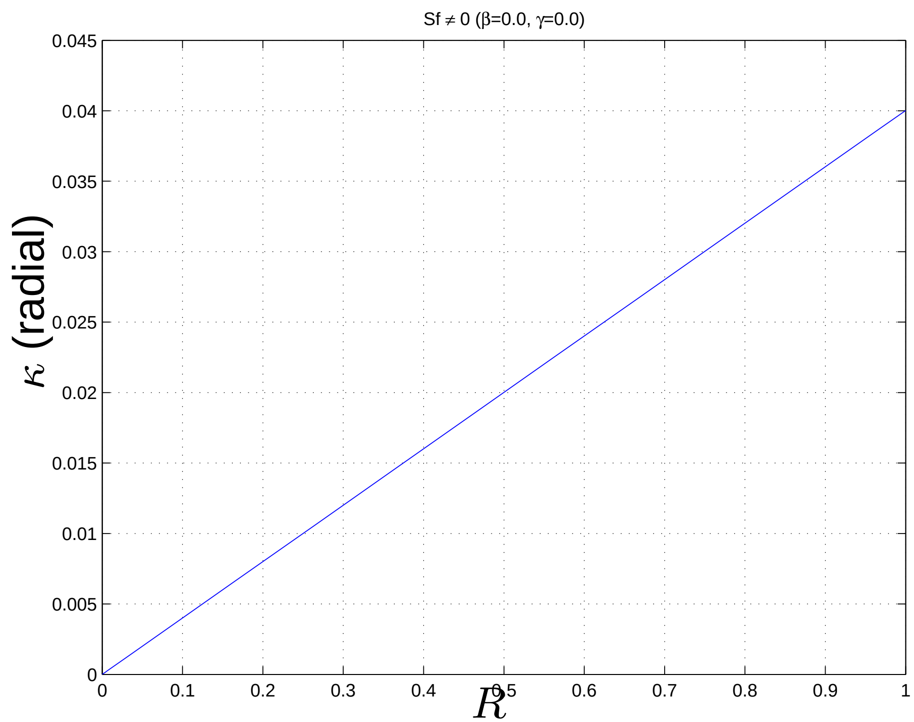
$$\boldsymbol{K}^{\mathrm{r}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{t}^{*} = \mathbf{0}\ \mathrm{Pa}$$

Remodelling Examples



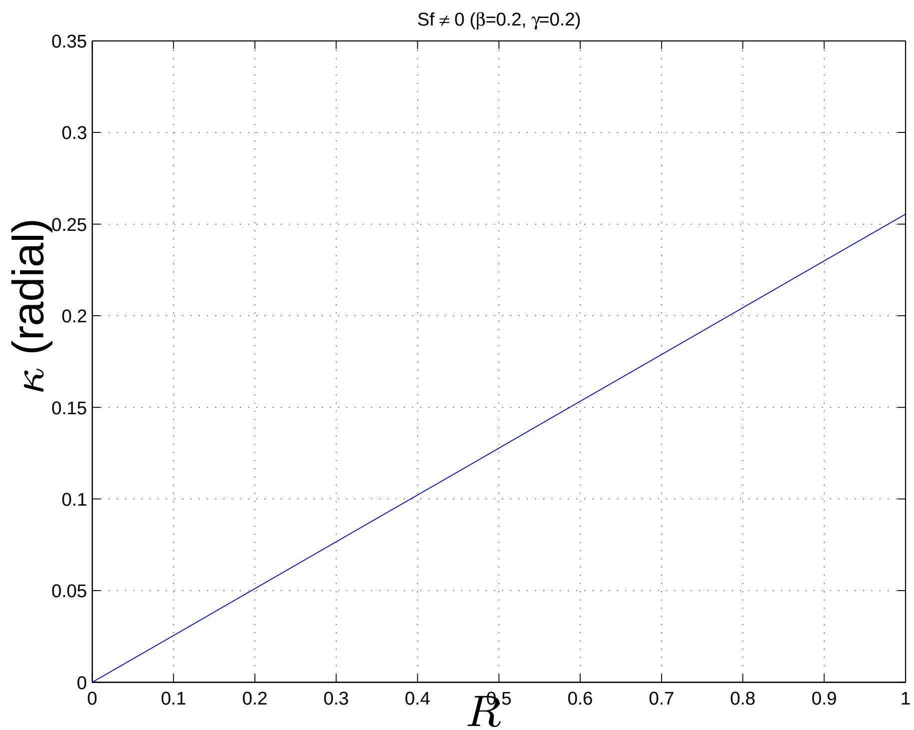
$$\boldsymbol{K}^{\mathrm{r}} = \begin{bmatrix} 1+\beta & 0 & 0 \\ 0 & 1+\gamma & 0 \\ 0 & 0 & 1+\gamma \end{bmatrix}, \beta = 0.2,\ \gamma = 0.2 - 0.6; \quad \boldsymbol{t}^* = \mathbf{0}\ \mathrm{Pa}$$

Remodelling Examples



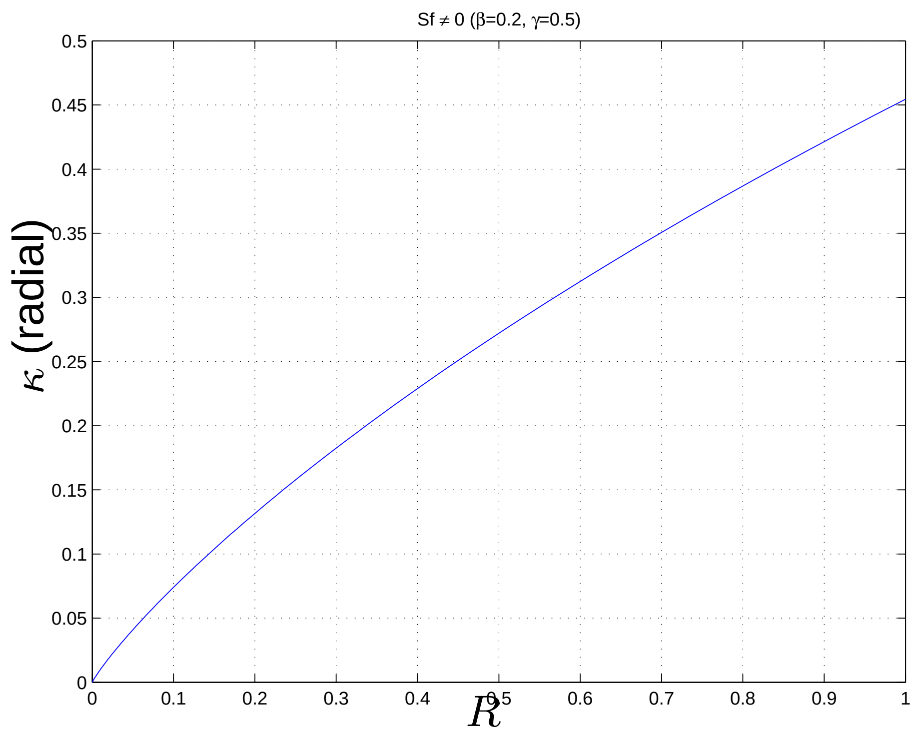
$$\boldsymbol{K}^{\mathrm{r}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{t}^{*} \approx 10^{9} \boldsymbol{e}_R \ \mathrm{Pa}$$

Remodelling Examples



$$\boldsymbol{K}^{\mathrm{r}} = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1.2 \end{bmatrix}, \quad \boldsymbol{t}^{*} \approx 10^{9} \boldsymbol{e}_{R} \ \mathrm{Pa}$$

Remodelling Examples



$$\boldsymbol{K}^{\mathrm{r}} = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}, \quad \boldsymbol{t}^* \approx 10^9 \boldsymbol{e}_R \text{ Pa}$$

Remarks

- Remodelling is coupled with growth—separate treatment for conceptual clarity
- The remodelled configuration, $\boldsymbol{\kappa}$ depends upon $\hat{\psi}^*(\bullet, \boldsymbol{K}^{\mathrm{c}}, \bullet)$
- Remodelled configuration is assumed to be an equilibrium state
  - Perturb conditions—new equilibrium
- Self-assembly processes in materials are similarly described by minimizing the Gibbs free energy of the systems with respect to the configuration